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The general addition theorem and ladder operators

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Abstract

We establish a condition for obtaining differential ladder operators of functions that satisfy the general addition theorem. We show that trigonometric and hyperbolic functions fulfill this condition and give some properties and applications of these operators.

Keywords: Addition theorem; Ladder operators; Trigonometric and hyperbolic functions

1. Introduction

The general addition theorem (GAT) was proved by Euler [1] and consists a generalization of a previous result[2] as shown in Siegel's book [2]. This theorem gives the value of a function, $s(x)$, evaluated as the sum of two numbers, $\mu + \nu$, through the values $s(\mu)$ and $s(\nu)$. The simplest functions that fulfill an algebraic addition theorem are the linear and exponential[3] functions. All the trigonometry is supported by the addition theorems of the circle *sine* and *cosine*. The circle *sine* is a particular case of the generalized *sine*. The generalized *sine* satisfies the so-called Euler GAT. More sophisticate addition theorems are concerned with the Bessel functions, Legendre polynomials, spherical and solid [4] harmonics, Coulomb functions [5] and so on. The purpose of this work is to obtain differential ladder operators in terms of the GAT and to study their properties and applications. We find that only the trigonometric and hyperbolic functions admit first-order differential operators of two parameters from the GAT. The investigation of ladder operators for the trigonometric functions has been carried out by several methods [6–8].

This work is organized as follows. In Section 2 we present the GAT and establish a condition for the existence of first-order ladder operators for functions that fulfill this theorem. In Section 3 we give the corresponding operators and study some properties of trigonometric functions. Some identities are obtained analytically. Section 4 is devoted

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2 It should be noted that this result was obtained by Fagnano.
3 This theorem has been generalized by Suslov [3].

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to obtaining the ladder operators for the Chebyshev functions directly from these operators. Finally, some conclusions are given in Section 5.

2. GAT and ladder operators

In 1768, Euler showed the GAT for the generalized *sine*, $w = s(t)$, which is the inverse function of the integral given by [1]

$$t = \int_0^w \frac{\mathrm{d}z}{\sqrt{1+\overline{m}z^2+\overline{n}z^4}}; \quad \overline{m}, \overline{n} \in R, \tag{1}$$

where henceforth the sign of square root is taken as "+". This theorem affirms that

$$s(\mu+\nu) = F[s(\mu), s(\nu)]; \quad \mu, \nu \in C, \tag{2}$$

where $s(\mu+\nu)$ can be explicitly expressed as

$$s(\mu+\nu) = \frac{s(\nu)}{1-\overline{n}s^2(\mu)s^2(\nu)}\sqrt{1+\overline{m}s^2(\mu)+\overline{n}s^4(\mu)} + \frac{\sqrt{1+\overline{m}s^2(\nu)+\overline{n}s^4(\nu)}}{1-\overline{n}s^2(\mu)s^2(\nu)}s(\mu), \tag{3}$$

from which we can obtain the addition theorems for functions $\sin(t)$, $\sinh(t)$, $sl(t)$ and $sn(t,k)$. The parameters $\overline{m}$ and $\overline{n}$ are taken as different values for different functions [1,2,10], e.g.

$$(\overline{m}, \overline{n}) = \begin{cases} \overline{m} = -1, \overline{n} = 0, & \text{for circle sine,} \\ \overline{m} = 1, \overline{n} = 0, & \text{for hyperbolic sine,} \\ \overline{m} = 0, \overline{n} = -1, & \text{for lemniscate } sl(t), \\ \overline{m} = -(1+k^2), \overline{n} = k^2, \quad k \in (0,1), & \text{for elliptic Jacobi } sn(t,k). \end{cases} \tag{4}$$

Generally speaking, we can express the $s(\mu x + \nu x)$ as

$$s(\mu x + \nu x) = f(x)\frac{1}{\mu}\frac{\mathrm{d}s(\mu x)}{\mathrm{d}x} - W(x)s(\mu x). \tag{5}$$

By comparing Eq. (3) ($\mu \to \mu x, \nu \to \nu x$) and Eq. (5), we can identify

$$W(x) = -\frac{\sqrt{1+\overline{m}s^2(\nu x)+\overline{n}s^4(\nu x)}}{1-\overline{n}s^2(\mu x)s^2(\nu x)}, \qquad f(x) = \frac{s(\nu x)}{1-\overline{n}s^2(\mu x)s^2(\nu x)}, \tag{6}$$

$$\frac{1}{\mu}\frac{\mathrm{d}s(\mu x)}{\mathrm{d}x} = \sqrt{1+\overline{m}s^2(\mu x)+\overline{n}s^4(\nu x)}, \tag{7}$$

where Eq. (6) are two simple definitions, but Eq. (7) is a required condition because the differential operator given in Eq. (5) must be a ladder operator for the function $s(t)$. Thus, we note that only the trigonometric and hyperbolic *sine* functions satisfy this condition (7).

3. Ladder operators for trigonometric functions

In this case we have $s(x) = \sin(\mu x)$, $\overline{m} = -1$ and $\overline{n} = 0$. Since Eq. (7) is obviously satisfied, substitution of Eq. (6) into Eq. (5) leads to the following μ, ν-dependent ladder operator:

$$A(\mu,\nu) \equiv \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x} - W(x)\right) = \sin(\nu x)\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}x} + \cos(\nu x), \quad \mu \neq 0 \tag{8}$$

from which we have $A(\mu, \nu = 0) = 1$, $A(\mu, -\nu) = A(-\mu, \nu)$.

Let us study some properties of these ladder operators. By considering real numbers μ and ν and acting with $A(\mu, \pm\nu)$ on the function

$$\psi_\mu(x) = K\sin(\mu x), \quad K = \text{constant}, \tag{9}$$

we obtain the following function:

$$\psi_{\mu\pm\nu}(x) = A(\mu, \pm\nu)\psi_\mu(x), \tag{10}$$

where the continuous parameter ν corresponds to the step of the ladder operator. For a given μ, it is shown that the operator $A(\mu, \nu)$ and the l-th-order differential operator

$$A\left(\mu + \sum_{i=1}^{l} \nu_{i-1}, \nu_l\right) \cdots A(\mu + \nu_1 + \nu_2, \nu_3)A(\mu + \nu_1, \nu_2)A(\mu, \nu_1) \equiv \prod_{j=1}^{l} A\left(\mu + \sum_{i=1}^{j} \nu_{i-1}, \nu_j\right), \tag{11}$$

applied to $\psi_\mu(x)$ generate the function $\psi_{\mu+\nu}(x)$, where $\nu_0 = 0$ and $\{\nu_i\}_{i=1}^{l}$ are arbitrary real numbers that satisfy

$$\sum_{i=1}^{l} \nu_i = \nu; \quad l = 2, 3, \ldots. \tag{12}$$

In fact, we can demonstrate that

$$A(\mu, \nu)\psi_\mu(x) = \prod_{j=1}^{l} A\left(\mu + \sum_{i=1}^{j} \nu_{i-1}, \nu_j\right)\psi_\mu(x) \equiv \psi_{\mu+\nu}(x). \tag{13}$$

Thus, we may define

$$A(\mu, \nu) = \prod_{j=1}^{l} A\left(\mu + \sum_{i=1}^{j} \nu_{i-1}, \nu_j\right), \tag{14}$$

where we assume that the equality "=" is true if and only if the operators act on $\psi_\mu(x)$ in the order given by Eq. (11). For instance, in the special case $\nu = 0$, Eq. (12) can be reduced to $\nu_1 + \nu_2 = 0$. For $\nu_1 = \eta \neq 0$, we obtain

$$A(\mu, 0)\psi_\mu(x) = A(\mu + \eta, -\eta)A(\mu, \eta)\psi_\mu(x) \equiv \psi_\mu(x), \tag{15}$$

from which, by considering Eq. (8), we have

$$A(\mu + \eta, -\eta)A(\mu, \eta) = -\frac{\sin^2(\eta x)}{\mu(\mu + \eta)}\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \mu(\mu + \eta)\cot^2(\eta x) - \mu\eta\right], \tag{16}$$

which can be easily verified by using Eq. (9).

Similarly, we have

$$A(\mu, \pm\nu)\widetilde{\psi}_\mu(x) = \widetilde{\psi}_{\mu\pm\nu}(x); \quad \mu \neq 0, \tag{17}$$

where

$$\widetilde{\psi}_\mu(x) = \widetilde{K}\cos(\mu x), \quad \widetilde{K} = \text{constant}. \tag{18}$$

Essentially, the ladder operators of $\psi_\mu(x)$ are the same as those of $\widetilde{\psi}_\mu(x)$.

We now turn to finding a function $\Psi(x)$ such that

$$A(\mu, \nu)\Psi(x) = 0. \tag{19}$$

From Eq. (8), Eq. (19) can be further re-expressed as

$$\left(\sin(\nu x)\frac{1}{\mu}\frac{\mathrm{d}}{\mathrm{d}x} + \cos(\nu x)\right)\Psi(x) = 0, \tag{20}$$

from which we have

$$\Psi(x) = c(\sin \nu x)^{-\mu/\nu}, \quad c = \text{constant}. \tag{21}$$

Since the function $\Psi(x)$ is in the set $\{\psi_\nu(x)\}_{\nu\in R}$, we find that if the condition $\nu = -\mu$ is applied, Eq. (21) can be identified as $\psi_\mu(x)$. Therefore, we have

$$A(\mu, -\mu)\psi_{-\mu}(x) = A(-\mu, \mu)\psi_\mu(x) = 0, \tag{22}$$

which can be seen as an adequate operator acting on any element of set $\{\psi_\nu(x)\}_{\nu\in R}$ connects it with zero. Otherwise, there does not exist an operator connecting an element in $\{\widetilde{\psi}_\nu(x)\}_{\nu\in R}$ with zero.

We now briefly study the trigonometric identities and some matrix elements. We know that $\psi_m(x)$ and $\widetilde{\psi}_m(x)$ in the interval $[x_0, x_0 + 2\pi]$ are linearly independent solutions of the Sturm–Liouville problem

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f_m(x) + m^2 f_m(x) = 0, \quad m = \pm 1, \pm 2, \ldots. \tag{23}$$

It is shown from Eq. (8) that

$$\cos(nx) = \frac{1}{2}[A(m,n) + A(m,-n)], \qquad \sin(nx)\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{2}[A(m,n) - A(m,-n)], \tag{24}$$

from which, together with Eq. (9), we have

$$\cos(nx)\sin(mx) = \frac{1}{2}[\sin(mx + nx) + \sin(mx - nx)]. \tag{25}$$

Similarly, we can generalize this result to more complicated case. Defining the operator $A = \sum_{k=-n}^{n} A(m,k)$ and acting with it on $\psi_m(x)$ allows us to obtain

$$A\psi_m(x) = \sum_{k=-n}^{n} A(m,k)\psi_m(x) = \sum_{k=-n}^{n} \psi_{m+k}(x) = K\sum_{k=-n}^{n} \sin(mx + kx). \tag{26}$$

Moreover, from Eq. (24) we have $A = 2\sum_{k=1}^{n}\cos(kx) + 1$ ($n \geq 1$). Acting with this on $\psi_m(x)$, we obtain

$$A\psi_m(x) = K\left[2\sum_{k=1}^{n}\cos(kx) + 1\right]\sin(mx). \tag{27}$$

Comparing this with Eq. (26) leads to

$$\sum_{k=-n}^{n}\sin(mx + kx) = \left[2\sum_{k=1}^{n}\cos(kx) + 1\right]\sin(mx). \tag{28}$$

If we write

$$\frac{1}{K}\psi_m(x) \to \langle x|m\rangle; \qquad \frac{1}{\widetilde{K}}\widetilde{\psi}_m(x) \to \langle x|m\rangle_\sim, \tag{29}$$

then from Eq. (24) we directly get

$$\cos(nx)|m\rangle = \frac{1}{2}[|m+n\rangle + |m-n\rangle]. \tag{30}$$

Thus, we are able to obtain the following result:

$$\langle m'|\cos(nx)|m\rangle = \frac{\pi}{2}[\delta_{m',m+n} + \delta_{m',m-n}], \quad m' \neq 0. \tag{31}$$

Repeating this process we may obtain

$$\cos^k(nx)|m\rangle = \frac{\pi}{2^k}\left[\sum_{j=0}^{k} c_j A(m+kn, -2jn)\right]|m+kn\rangle, \quad c_j = \begin{cases} 1; & j = 0, k, \\ k; & j = 1, \ldots, k-1, \end{cases} \tag{32}$$

and

$$\langle m'|\cos^k(nx)|m\rangle = \frac{\pi}{2^k}\left[\sum_{j=0}^{k} c_j \delta_{m',m+kn-2jn}\right]. \tag{33}$$

Likewise, we can obtain

$$\cos^k(nx)|m\rangle_\sim = \frac{\pi}{2^k}\left[\sum_{j=0}^{k} c_j A(m+kn,-2jn)\right]|m+kn\rangle_\sim, \tag{34}$$

and

$$_\sim\langle m'|\cos^k(nx)|m\rangle_\sim = \frac{\pi}{2^k}\left[\sum_{j=0}^{k} c_j \delta_{m',m+kn-2jn}\right]. \tag{35}$$

4. Ladder operators for Chebyshev functions

Let us study the ladder operators $A(m,\pm 1)$. Such operators were identified as elements of an $su(1,1)$ algebra for the infinitely deep square-well potential [8]. They were also used to study the temporally stable coherent states for infinite well and Pösch–Teller potentials [9].

If the ladder operators for $\psi_m(x)$ and $\widetilde{\psi}_m(x)$ are taken as

$$A(m,\pm 1)(x) = \pm\sin(x)\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}x} + \cos(x), \tag{36}$$

then we have

$$A(m,\pm 1)(y) = \mp(1-y^2)\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}y} + y, \quad y = \cos x,\, y \in [-1,1]. \tag{37}$$

Thus, Eq. (23) can be rewritten as

$$(1-y^2)\frac{\mathrm{d}^2}{\mathrm{d}y^2}F_m(y) - y\frac{\mathrm{d}}{\mathrm{d}y}F_m(y) + m^2 F_m(y) = 0, \tag{38}$$

whose solutions are given by [11]

$$F_m(y) = \begin{cases} T_m(y) \equiv \widetilde{K}\cos(m\cos^{-1} y) = \widetilde{\psi}_m(x(y)), \\ V_m(y) \equiv K\sin(m\cos^{-1} y) = \psi_m(x(y)). \end{cases} \tag{39}$$

Eq. (38) is the Chebyshev differential equation of type I. To obtain the ladder operators directly for $F_m(y)$, we consider

$$A(m,\pm 1)(x)\begin{pmatrix}\widetilde{\psi}_m(x)\\ \psi_m(x)\end{pmatrix} = \begin{pmatrix}\widetilde{\psi}_{m\pm 1}(x)\\ \psi_{m\pm 1}(x)\end{pmatrix}, \tag{40}$$

from which, together with Eqs. (37) and (39), we have

$$A(m,\pm 1)(y)\begin{pmatrix}T_m(y)\\ V_m(y)\end{pmatrix} = \left(\mp(1-y^2)\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}y} + y\right)\begin{pmatrix}T_m(y)\\ V_m(y)\end{pmatrix} = \begin{pmatrix}T_{m\pm 1}(y)\\ V_{m\pm 1}(y)\end{pmatrix}, \quad m \neq 0, \tag{41}$$

from which we can identify the ladder operators for the Chebyshev polynomials of type I.

The solutions $U_n(y)$ and $W_n(y)$ of the Chebyshev equation of type II are closely related to those of type I as [11]

$$V_{n+1}(y) = (1-y^2)^{1/2}U_n(y), \tag{42}$$

$$T_{n+1}(y) = (1-y^2)^{1/2}W_n(y). \tag{43}$$

From Eqs. (41) and (42), we obtain

$$A(n+1,-1)(y)V_{n+1}(y) = \begin{cases} V_n(y) = (1-y^2)^{1/2}U_{n-1}(y), \\ \dfrac{(1-y^2)^{1/2}}{n+1}\left[(1-y^2)\dfrac{\mathrm{d}}{\mathrm{d}y} + ny\right]U_n(y), \end{cases} \tag{44}$$

from which we have

$$\frac{1}{n+1}\left[(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y} + ny\right]U_n(y) = U_{n-1}(y). \tag{45}$$

Similarly, we can show that

$$\frac{1}{n+1}\left[-(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y} + (n+2)y\right]U_n(y) = U_{n+1}(y). \tag{46}$$

Thus, we can obtain the ladder operators for the Chebyshev functions of type II from Eqs. (45) and (46). A similar procedure allows us to obtain those for $W_n(y)$.

5. Conclusions

We have defined in a consistent form μ, ν-dependent first-order differential ladder operators for functions that satisfy the GAT and Eq. (7). We find that only the trigonometric and hyperbolic functions satisfy this condition. Some trigonometric identities and analytical matrix elements have been also presented. We have studied some properties of these operators and obtained ladder operators for the Chebyshev functions.

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References

[1] L. Euler, Institutiones Calculi Integralis, Volumen Primum, 1768. Reprint as Leonhardi Euleri Opera Omnia, Series I, Opera Mathematica, Volumen XI, Engel et Schlesinger, eds., B. G. Teubneri Leipzig, 1913.
[2] C.L. Siegel, Topics in Complex Function Theory, in: Elliptic Functions and Uniformization Theory (Shenitzer, Solitar, Trans.), vol. I, Wiley-Interscience, New York, 1969.
[3] S. Suslov, J. Phys. A 33 (2000) L375.
[4] M.J. Caola, J. Phys. A 11 (2) (1978) L23.
[5] M. Pauli, K. Alder, J. Phys. A 9 (6) (1976) 905.
[6] D. Aebersold, A. Langerman, Amer. J. Phys. 45 (3) (1977) 272.
[7] S. Kais, R.D. Levine, Phys. Rev. A 34 (6) (1986) 4615.
[8] S.H. Dong, Z.Q. Ma, Amer. J. Phys. 70 (5) (2002) 520.
[9] J.P. Antoine, et al., J. Math. Phys. 42 (6) (2001) 2349 and references therein.
[10] A.I. Markushevich, The Remarkable Sine Functions, Amer. Elsevier, New York, 1966.
[11] T.J. Rivlin, Chebyshev Polynomials from Approximation Theory to Algebra and Number Theory, second edn, in: Pure and Applied Mathematics a Wiley-Interscience series of texts, monographs and Tracts, 1990.